

# N-derivations for finitely generated graded Lie algebras

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**Abstract:**  $N$ -derivation is the natural generalization of derivation and triple derivation. Let  $\mathcal{L}$  be a finitely generated Lie algebra graded by a finite dimensional Cartan subalgebra. In this paper, a sufficient condition for Lie  $N$ -derivation algebra of  $\mathcal{L}$  coinciding with Lie derivation algebra of  $\mathcal{L}$  is given. As applications, any  $N$ -derivation of Schrödinger-Virasoro algebra, generalized Witt algebras, Kac-Moody algebras and their Borel subalgebras, is a derivation.

**Keywords:** Derivation, Cartan subalgebra, Virasoro algebra, Kac-Moody algebra

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## 1 Introduction

Let  $\mathcal{L}$  be a Lie algebra over an arbitrary field  $F$ . Recall that an  $F$ -linear mapping  $\psi : \mathcal{L} \rightarrow \mathcal{L}$  is called a Lie derivation of  $\mathcal{L}$  if

$$\psi([x, y]) = [\psi(x), y] + [x, \psi(y)], \quad \forall x, y \in \mathcal{L}.$$

Let  $N \geq 2$  be a positive integer. We say  $\varphi$  is an  $N$ -derivation of  $\mathcal{L}$ , if  $\varphi$  is a linear map from  $\mathcal{L}$  to itself, and satisfies

$$\varphi([x_1, \dots, x_{N-1}, x_N]) = \sum_{i=1}^N [x_1, \dots, x_{i-1}, \varphi(x_i), x_{i+1}, \dots, x_N] \quad (\forall x_1, \dots, x_N \in \mathcal{L}),$$

where  $[x_1, \dots, x_{N-1}, x_N] = [x_1 \cdots [x_{N-2}, [x_{N-1}, x_N]] \cdots]$ .

The set of Lie  $N$ -derivation is clearly a Lie algebra under the usual bracket and will be denoted by  $\text{Der}^{(N)}\mathcal{L}$ . By the definition, we have  $\text{Der}^{(2)}\mathcal{L} = \text{Der}\mathcal{L}$ , where  $\text{Der}\mathcal{L}$

is the Lie derivation algebra of  $\mathcal{L}$ . A Lie derivation is obviously a Lie  $N$ -derivation, which implies  $\text{Der}\mathcal{L} \subseteq \text{Der}^{(N)}\mathcal{L}$ . In general, the derivation algebra  $\text{Der}\mathcal{L}$  is a proper subalgebra of  $\text{Der}^{(N)}\mathcal{L}$  for  $N \geq 3$ . It would be interesting to know when these two algebras coincide.

For  $N = 3$ , Lie  $N$ -derivation is called Lie triple derivation which has received a fair amount of attentions (see [4, 6, 8, 12, 13, 14]). In [13], Wang and Yu showed that a linear map on Borel subalgebra of finite-dimensional simple Lie algebra over an algebraically closed field  $F$  of characteristic zero is a Lie triple derivation, if and only if it is an inner derivation. Which are examples for Lie triple derivation algebra coinciding with Lie derivation algebra, and the Lie algebras concerned are graded by a finite-dimensional Cartan subalgebra.

The Schrödinger-Virasoro algebra  $\mathfrak{sv}$  introduced by Henkel in [3], during his study on the invariance of the free Schrödinger equation, is an infinite-dimensional Lie algebra with  $\mathbb{C}$ -basis  $\{L_n, M_n, Y_{n+\frac{1}{2}}, C \mid n \in \mathbb{Z}\}$  subject to the following Lie brackets:

$$\begin{aligned} [L_m, L_n] &= (n-m)L_{n+m} + \delta_{m+n,0} \frac{n^3-n}{12} C, & [L_m, M_n] &= nM_{n+m}, \\ [L_m, Y_{n+\frac{1}{2}}] &= (n + \frac{1-m}{2})Y_{m+n+\frac{1}{2}}, & [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] &= (n-m)M_{m+n+1}, \\ [M_m, M_n] &= [M_m, Y_{n+\frac{1}{2}}] = 0, & [\mathfrak{sv}, C] &= \{0\}. \end{aligned}$$

Due to its important applications in many areas of Mathematics and Physics, the structure and representation theory of  $\mathfrak{sv}$  have been extensively studied (see [2, 7, 10, 11]). For example, in [10], Rosen and Unterberger presented detailed cohomological study and determined that  $\mathfrak{sv}$  has three linear independent outer derivations. This follows that not all derivation of  $\mathfrak{sv}$  is an inner derivation.

In this paper, we consider Lie  $N$ -derivations for finitely generated graded Lie algebra  $\mathcal{L}$ . In section 2, we prove that the Lie  $N$ -derivation algebra of  $\mathcal{L}$  is graded, which generalizes the result obtained by Farnsteiner (see [1]). In section 3, we assume  $\mathcal{L}$  is graded by a finite-dimensional Cartan subalgebra and is over an algebraically closed field  $F$  of characteristic zero. We show that  $\text{Der}^{(N)}\mathcal{L} = \text{Der}\mathcal{L}$  for  $N \geq 3$ , if  $\mathcal{L}$  satisfies condition (P) (see theorem 3.2). As applications, in section 4, we show that the Lie  $N$ -derivation algebras of Schrödinger-Virasoro algebra, generalized Witt algebras and Kac-Moody algebras are coinciding with the derivation algebras. In particular, any  $N$ -derivation of finite-dimensional semisimple complex Lie algebra is an inner derivation.

## 2 $N$ -derivation of finitely generated graded Lie algebra

Let  $G$  be an abelian group, and let  $\mathcal{L} = \bigoplus_{\alpha \in G} \mathcal{L}_\alpha$  be a  $G$ -graded Lie algebra. For  $x_1, \dots, x_n \in \mathcal{L}$ , set

$$[x_1, \dots, x_{n-1}, x_n] = [x_1 \cdots [x_{n-2}, [x_{n-1}, x_n]] \cdots].$$

Let  $N \geq 2$  be a positive integer. As above, denote  $\text{Der}^{(N)}(\mathcal{L})$ ,  $\text{Der}(\mathcal{L})$  the set of all  $N$ -derivations of  $\mathcal{L}$  and the set of all derivations of  $\mathcal{L}$ , respectively. We have  $\text{Der}(\mathcal{L}) \subseteq \text{Der}^{(N)}(\mathcal{L})$ . An  $N$ -derivation  $\varphi$  is called an  $N$ -derivation of homogeneous degree  $\alpha$  if  $\varphi(\mathcal{L}_\beta) \subseteq \mathcal{L}_{\alpha+\beta}$ , for  $\alpha, \beta \in G$ . Set

$$\text{Der}_\alpha^{(N)}(\mathcal{L}) = \{\varphi \in \text{Der}^{(N)}(\mathcal{L}) \mid \deg \varphi = \alpha\}.$$

**Lemma 2.1** *Let  $G$  be an abelian group. For any finitely generated  $G$ -graded Lie algebra  $\mathcal{L} = \bigoplus_{\alpha \in G} \mathcal{L}_\alpha$ , we have*

$$\text{Der}^{(N)}(\mathcal{L}) = \bigoplus_{\alpha \in G} \text{Der}_\alpha^{(N)}(\mathcal{L}).$$

*Proof* For  $\alpha \in G$ , let  $\rho_\alpha : \mathcal{L} \rightarrow \mathcal{L}_\alpha$  denote the canonical projection. Let  $S$  be a finite subset generating  $\mathcal{L}$ . Set

$$Y := S \cup \{[x_1, \dots, x_m] \mid x_1, \dots, x_m \in S, m = 2, \dots, N-1\},$$

$Y$  is also a finite subset of  $\mathcal{L}$ . Clearly,  $Y$  generates  $\mathcal{L}$  as an  $N$ -Lie algebra (i.e.,  $\mathcal{L}$  is the smallest subspace containing  $Y$  and stable under taking iterated brackets of the form  $[x_1, \dots, x_N]$ ). For  $\varphi \in \text{Der}^{(N)}(\mathcal{L})$ , there is a finite set  $K \subseteq G$  such that

$$Y \cup \varphi(Y) \subset \sum_{\alpha \in K} \mathcal{L}_\alpha. \quad (1)$$

For  $\alpha \in G$ , set  $\varphi_\alpha := \sum_{\beta \in G} \rho_{\alpha+\beta} \varphi \rho_\beta$ . Since for  $x_{\beta_i} \in \mathcal{L}_{\beta_i}$  ( $i = 1, \dots, N$ ), we have

$$\begin{aligned} & \varphi_\alpha([x_{\beta_1}, \dots, x_{\beta_N}]) \\ &= \rho_{\alpha+\beta_1+\dots+\beta_N} \varphi([x_{\beta_1}, \dots, x_{\beta_N}]) \\ &= \rho_{\alpha+\beta_1+\dots+\beta_N} \left( \sum_{i=1}^N [x_{\beta_1}, \dots, x_{\beta_{i-1}}, \varphi(x_{\beta_i}), x_{\beta_{i+1}}, \dots, x_{\beta_N}] \right) \\ &= \sum_{i=1}^N [x_{\beta_1}, \dots, x_{\beta_{i-1}}, \rho_{\alpha+\beta_i} \varphi(x_{\beta_i}), x_{\beta_{i+1}}, \dots, x_{\beta_N}] \\ &= \sum_{i=1}^N [x_{\beta_1}, \dots, x_{\beta_{i-1}}, \varphi_\alpha(x_{\beta_i}), x_{\beta_{i+1}}, \dots, x_{\beta_N}], \end{aligned}$$

which follows that  $\varphi_\alpha \in \text{Der}_\alpha^{(N)}(\mathcal{L})$ .

Let  $T := \{\alpha - \beta \mid \alpha, \beta \in K\}$ , then  $T$  is finite. For  $y \in Y$ , we obtain

$$\begin{aligned} \varphi(y) &\stackrel{(a)}{=} \sum_{\alpha, \beta \in K} \rho_\alpha \varphi \rho_\beta(y) \\ &= \sum_{\alpha, \beta \in K} \rho_{\alpha-\beta+\beta} \varphi \rho_\beta(y) \\ &= \sum_{\beta \in K} \sum_{\gamma \in K-\beta} \rho_{\gamma+\beta} \varphi \rho_\beta(y) \\ &\stackrel{(b)}{=} \sum_{\beta \in K} \sum_{\gamma \in T} \rho_{\gamma+\beta} \varphi \rho_\beta(y) \\ &= \sum_{\gamma \in T} \sum_{\beta \in K} \rho_{\gamma+\beta} \varphi \rho_\beta(y) \\ &\stackrel{(c)}{=} \sum_{\gamma \in T} \sum_{\beta \in G} \rho_{\gamma+\beta} \varphi \rho_\beta(y) \\ &= \sum_{\gamma \in T} \varphi_\gamma(y), \end{aligned}$$

where (a), (b) and (c) follow from (1). This shows that the Lie  $N$ -derivation  $\varphi$  and  $\sum_{\gamma \in T} \varphi_\gamma$  coincide on  $Y$ . By the construction of  $Y$  and the definition of Lie  $N$ -derivation, we obtain  $\varphi = \sum_{\gamma \in T} \varphi_\gamma$ .  $\square$

**Remark 2.2** In case  $N = 2$ ,  $\text{Der}^{(N)}(\mathcal{L}) = \text{Der}(\mathcal{L})$ , the above lemma is contained in the proposition 1.1 of [1]. In what follows, we assume  $N \geq 3$ .

### 3 Main theorem

Let  $\mathcal{L}$  be a finitely generated Lie algebra graded by a nontrivial finite dimensional Cartan subalgebra  $\mathcal{H}$ . Let  $\mathcal{H}^*$  be the dual space of  $\mathcal{H}$ . For  $\alpha \in \mathcal{H}^*$ ,

$$\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\}$$

is the root space associated to  $\alpha$ . Let  $R = \{\alpha \in \mathcal{H}^* \mid \mathcal{L}_\alpha \neq \{0\}\}$ ,  $R$  is called the root system of  $\mathcal{L}$  with respect to  $\mathcal{H}$ . Thus we have

$$\mathcal{L} = \bigoplus_{\alpha \in R} \mathcal{L}_\alpha, \quad \text{and} \quad \mathcal{L}_0 = \mathcal{H}.$$

Set  $R^\times = \{\alpha \in R \mid \alpha \neq 0\}$ ,  $R_\pm = \{\alpha \in R^\times \mid -\alpha \in R\}$ .

**Definition 3.1** We say  $\mathcal{L}$  satisfying property (P) if for  $\alpha \in R_\pm$ , every element  $x_\alpha \in \mathcal{L}_\alpha$  satisfies (P1) or (P2), where

(P1)  $x_\alpha = [y_{\alpha-\beta}, y_\beta]$  for some  $y_{\alpha-\beta} \in \mathcal{L}_{\alpha-\beta}$  and  $y_\beta \in \mathcal{L}_\beta$ , with  $\beta \notin \{0, \alpha\}$ .

(P2)  $[x_\alpha, x_\alpha, x_{-\alpha}] \neq 0$  for some  $x_{-\alpha} \in \mathcal{L}_{-\alpha}$ .

**Theorem 3.2** Let  $\mathcal{L}$  be a finitely generated Lie algebra with a nontrivial finite dimensional Cartan subalgebra  $\mathcal{H}$ , and let  $N \geq 3$  be a positive integer. If  $N$  is even or if  $\mathcal{L}$  satisfying property (P), then we have

$$\text{Der}^{(N)}(\mathcal{L}) = \text{Der}(\mathcal{L}).$$

*Proof* Assume  $\mathcal{H}, R$  as above, let  $Q = \mathbb{Z}R$  be the abelian group. Clearly,  $\mathcal{L}$  is a finitely generated  $Q$ -graded Lie algebra. Using lemma 2.1, we have

$$\text{Der}^{(N)}(\mathcal{L}) = \bigoplus_{\alpha \in Q} \text{Der}_\alpha^{(N)}(\mathcal{L}).$$

Now let  $\varphi \in \text{Der}_\gamma^{(N)}(\mathcal{L})$ ,  $\gamma \in Q$ . In what follows we will prove  $\varphi$  is a derivation. We divide the argument into two cases.

**Case one:**  $\gamma = 0$ .

For  $x, y \in \mathcal{H}$ , we have  $\varphi([x, y]) = 0 = [\varphi(x), y] + [x, \varphi(y)]$  as required. Suppose  $0 \neq y \in \mathcal{L}_\alpha$  for some  $\alpha \in R^\times$ , then there is  $h_\alpha \in \mathcal{H}$  such that  $\alpha(h_\alpha) = 1$ . Taking  $\varphi$  on

$$y = \underbrace{[h_\alpha, \dots, h_\alpha]}_{N-1}, y],$$

we have  $\varphi(y) = (N-1)\alpha(\varphi(h_\alpha))y + \varphi(y)$ , which implies  $\alpha(\varphi(h_\alpha)) = 0$ . Thus, for  $x \in \mathcal{L}$ , taking  $\varphi$  on

$$[x, y] = [x, \underbrace{h_\alpha, \dots, h_\alpha}_{N-2}, y],$$

we have  $\varphi([x, y]) = [\varphi(x), y] + [x, \varphi(y)]$  as required.

**Case two:**  $\gamma \neq 0$ .

Since  $\gamma \neq 0$ , there exists  $h_\gamma \in \mathcal{H}$ , such that  $\gamma(h_\gamma) = 1$ . For  $h \in \mathcal{H}$ , taking  $\varphi$  on

$$0 = [\underbrace{h_\gamma, \dots, h_\gamma}_{N-1}, h],$$

we have  $0 = [\varphi(h_\gamma), h] + \varphi(h)$ , that is  $\varphi(h) = (-\text{ad}\varphi(h_\gamma))(h)$ . Set  $\psi = \varphi + \text{ad}\varphi(h_\gamma)$ ,  $\psi$  is an  $N$ -derivation and  $\psi(\mathcal{H}) = \{0\}$ . In what follows, we use two subcases to show

$$\psi(x_\alpha) = 0, \quad \text{for } 0 \neq x_\alpha \in \mathcal{L}_\alpha, \alpha \in R^\times.$$

Thus  $\varphi = -\text{ad}\varphi(h_\gamma)$  is a derivation.

**Subcase one:**  $N$  is even or  $\gamma \neq -2\alpha$ .

Since  $N$  is even or  $\gamma \neq -2\alpha$ , there is  $\bar{h} \in \mathcal{H}$  such that  $\alpha(\bar{h})^{N-1} \neq (\alpha + \gamma)(\bar{h})^{N-1}$ . Taking  $\psi$  on

$$\alpha(\bar{h})^{N-1}x_\alpha = [\underbrace{\bar{h}, \dots, \bar{h}}_{N-1}, x_\alpha],$$

we have  $\alpha(\bar{h})^{N-1}\psi(x_\alpha) = (\alpha + \gamma)(\bar{h})^{N-1}\psi(x_\alpha)$ , which implies  $\psi(x_\alpha) = 0$ , as required.

**Subcase two:**  $N$  is odd and  $\gamma = -2\alpha$ .

Since  $\alpha \neq 0$ , then there is  $h_\alpha \in \mathcal{H}$  such that  $\alpha(h_\alpha) = 1$ . If  $\mathcal{L}_{-\alpha} = \{0\}$ , then we have  $\psi(x_\alpha) = 0$ , as required. Suppose  $\mathcal{L}_{-\alpha} \neq \{0\}$ ,  $x_\alpha \in \mathcal{L}_\alpha$ . If  $x_\alpha$  satisfies (P1), then  $x_\alpha = [y_{\alpha-\beta}, y_\beta]$  for some  $y_{\alpha-\beta} \in \mathcal{L}_{\alpha-\beta}$  and  $y_\beta \in \mathcal{L}_\beta$ , with  $\beta \notin \{0, \alpha\}$ . Using subcase one and taking  $\psi$  on

$$x_\alpha = [\underbrace{h_\alpha, \dots, h_\alpha}_{N-2}, y_{\alpha-\beta}, y_\beta],$$

then we have  $\psi(x_\alpha) = 0$ . If  $x_\alpha$  satisfies (P2), then  $[x_\alpha, x_\alpha, x_{-\alpha}] \neq 0$  for some  $x_{-\alpha} \in \mathcal{L}_{-\alpha}$ . Firstly, using subcase one, we have  $\psi(x_{-\alpha}) = 0$ . Next, taking  $\psi$  on

$$0 = [\underbrace{h_\alpha, \dots, h_\alpha}_{N-2}, x_{-\alpha}, x_\alpha],$$

then we have  $(-2)^{N-2}[x_{-\alpha}, \psi(x_\alpha)] = 0$ , which implies  $[x_{-\alpha}, \psi(x_\alpha)] = 0$ . Finally, since  $N$  is odd, setting  $h' = [x_\alpha, x_{-\alpha}]$  and taking  $\psi$  on

$$\alpha(h')^{N-2}x_\alpha = [\underbrace{h', \dots, h'}_{N-3}, x_\alpha, x_{-\alpha}, x_\alpha],$$

then we have

$$\begin{aligned}
& \alpha(h')^{N-2}\psi(x_\alpha) \\
&= \underbrace{[h', \dots, h', \psi(x_\alpha), x_{-\alpha}, x_\alpha]}_{N-3} \\
&= (-\alpha)(h')^{N-2}\psi(x_\alpha) \\
&= -\alpha(h')^{N-2}\psi(x_\alpha).
\end{aligned}$$

Since  $[x_\alpha, x_\alpha, x_{-\alpha}] \neq 0$ , which implies  $\alpha(h') \neq 0$ , we have  $\psi(x_\alpha) = 0$  as required.

Therefore, every  $N$ -derivations of homogeneous degree  $\gamma$  are derivations. This completes the proof.  $\square$

**Remark 3.3** If  $\mathcal{L}$  is a finitely generated Lie algebra graded by a nontrivial finite dimensional Cartan subalgebra  $\mathcal{H}$ , but doesn't satisfy the property (P), then  $\text{Der}^{(2N+1)}(\mathcal{L}) = \text{Der}(\mathcal{L})$  doesn't holds in general for  $N \geq 1$ . See example 4.1.3 in the following section.

## 4 Applications

### 4.1 Schrödinger-Virasoro algebra

The Schrödinger-Virasoro algebra  $\mathfrak{sv}$  is an infinite-dimensional Lie algebra with  $\mathbb{C}$ -basis  $\{L_n, M_n, Y_{n+\frac{1}{2}}, C \mid n \in \mathbb{Z}\}$  subject to the following Lie brackets:

$$\begin{aligned}
[L_m, L_n] &= (n-m)L_{n+m} + \delta_{m+n,0} \frac{n^3-n}{12} C, & [L_m, M_n] &= nM_{n+m}, \\
[L_m, Y_{n+\frac{1}{2}}] &= (n + \frac{1-m}{2})Y_{m+n+\frac{1}{2}}, & [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] &= (n-m)M_{m+n+1}, \\
[M_m, M_n] &= [M_m, Y_{n+\frac{1}{2}}] = 0, & [\mathfrak{sv}, C] &= \{0\}.
\end{aligned}$$

It is easy to see the following facts about  $\mathfrak{sv}$  :

- (i)  $\mathfrak{sv}$  is generated by  $L_1, L_2, L_{-2}, M_1$  and  $Y_{-\frac{1}{2}}$ .
- (ii)  $\mathfrak{sv} = \bigoplus_{p \in \frac{1}{2}\mathbb{Z}} \mathfrak{sv}_p$  is a  $\frac{1}{2}\mathbb{Z}$  graded Lie algebra according to the Cartan algebra  $\mathfrak{h} = \mathbb{C}L_0 \oplus \mathbb{C}M_0 \oplus \mathbb{C}C$ , where,  $\mathfrak{sv}_n = \mathbb{C}L_n \oplus \mathbb{C}M_n$  for  $n \in \mathbb{Z}^\times (= \mathbb{Z} \setminus \{0\})$ ,  $\mathfrak{sv}_{n+\frac{1}{2}} = \mathbb{C}Y_{n+\frac{1}{2}}$  for  $n \in \mathbb{Z}$ , and  $\mathfrak{sv}_0 = \mathfrak{h}$ .
- (iii)  $[L_n, L_{-n}, L_n] = -2n^2 L_n$ ,  $[L_{2n+1}, Y_{-n-\frac{1}{2}}] = (-2n-1)Y_{n+\frac{1}{2}}$ ,  $[L_{-2n-1}, M_{3n+1}] = (3n+1)M_n$ , for  $n \in \mathbb{Z}^\times$ .

Therefore,  $\mathfrak{sv}$  is a finitely generated Lie algebra graded by a nontrivial finite dimensional Cartan subalgebra  $\mathfrak{h}$  and satisfies property (P). Using theorem 3.2, we have corollary 4.1.1.

**Corollary 4.1.1**  $\text{Der}^{(N)}(\mathfrak{sv}) = \text{Der}(\mathfrak{sv})$  ( $N \geq 3$ ).

**Example 4.1.2** Let  $K = \text{span}_{\mathbb{C}}\{L_0, M_1, M_{-1}\}$  be the subalgebra of  $\mathfrak{sv}$ .  $K$  is a finitely generated Lie algebra with Cartan subalgebra  $\mathcal{H} = \mathbb{C}L_0$ , but doesn't satisfy property (P). Let  $\varphi : K \rightarrow K$  be a linear map such that

$$\varphi(L_0) = \varphi(M_{-1}) = 0, \quad \varphi(M_1) = M_{-1}.$$

Taking  $\varphi$  on  $[L_0, M_1] = M_1$ , we get  $\varphi$  is not a derivation of  $K$ . But one can check that  $\varphi$  is an  $N$ -derivation of  $K$  for any odd integer  $N \geq 3$ .

## 4.2 Generalised Witt algebra

Let  $\mathcal{A}$  be the Laurent polynomial ring  $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_d^{\pm 1}]$  with commuting variables, and let  $\mathfrak{w}_d$  be the Lie algebra of derivations of  $\mathcal{A}$ .  $\mathfrak{w}_d$  is called generalised Witt algebra, and is also the Lie algebra of diffeomorphisms of torus  $T^d$  (see [9]). When  $d = 1$ ,  $\mathfrak{w}_d$  is the Witt algebra and its universal central extension is called the Virasoro algebra. When  $d \geq 2$ ,  $\mathfrak{w}_d$  has no nontrivial central extension.

Let  $\mathbb{Z}^d = \mathbb{Z}\varepsilon_1 \oplus \dots \oplus \mathbb{Z}\varepsilon_d$ . For  $\mathbf{n} = n_1\varepsilon_1 + \dots + n_d\varepsilon_d \in \mathbb{Z}^d$ , let  $t^{\mathbf{n}} = t_1^{n_1}t_2^{n_2}\dots t_d^{n_d}$  and let  $D_j(\mathbf{n}) = t^{\mathbf{n}}t_j \frac{\partial}{\partial t_j}$ . Then  $\mathfrak{w}_d = \text{span}_{\mathbb{C}}\{D_j(\mathbf{n}) | \mathbf{n} \in \mathbb{Z}^d, j = 1, 2, \dots, d\}$  with the following Lie structure:

$$[D_j(\mathbf{n}), D_k(\mathbf{m})] = m_j D_k(\mathbf{n} + \mathbf{m}) - n_k D_j(\mathbf{n} + \mathbf{m}), \quad \forall \mathbf{n}, \mathbf{m} \in \mathbb{Z}^d, j, k = 1, 2, \dots, d.$$

It is easy to see the following facts about  $\mathfrak{w}_d$ :

- (i)  $\mathfrak{w}_d$  is generated by  $\{D_i(\pm\varepsilon_j), D_i(\pm 2\varepsilon_j) | i, j = 1, 2, \dots, d\}$ .
- (ii)  $\mathfrak{w}_d = \bigoplus_{\mathbf{n} \in \mathbb{Z}^d} (\mathfrak{w}_d)_{\mathbf{n}}$  is a  $\mathbb{Z}^d$ -graded Lie algebra respects to the Cartan subalgebra  $\mathcal{H} = \text{span}_{\mathbb{C}}\{D_1(\mathbf{0}), \dots, D_d(\mathbf{0})\}$ , where  $(\mathfrak{w}_d)_{\mathbf{n}} = \text{span}_{\mathbb{C}}\{D_1(\mathbf{n}), \dots, D_d(\mathbf{n})\}$ .
- (iii)  $[D_i(-(2n_i + 1)\varepsilon_i), D_i(\mathbf{n} + (2n_i + 1)\varepsilon_i)] = (5n_i + 2)D_i(\mathbf{n})$ .

Therefore,  $\mathfrak{w}_d$  is a finitely generated Lie algebra graded by a nontrivial finite dimensional Cartan subalgebra  $\mathcal{H}$  and satisfies property (P). Using theorem 3.2, we have corollary 4.2.

**Corollary 4.2**  $\text{Der}^{(N)}(\mathfrak{w}_d) = \text{Der}(\mathfrak{w}_d)$  for  $N \geq 3$ .

## 4.3 Kac-Moody algebra

Recall that a matrix  $A = (a_{ij})_{i,j=1}^n$  is called a generalized Cartan matrix if it satisfies the following conditions:

- (C1)  $a_{ii} = 2$  for  $i = 1, \dots, n$ ;
- (C2)  $a_{ij}$  are nonpositive integers for  $i \neq j$ ;
- (C3)  $a_{ij} = 0$  implies  $a_{ji} = 0$ .

Let  $A = (a_{ij})_{i,j=1}^n$  be a generalized Cartan matrix of rank  $l$ . A *realization* of  $A$  is a triple  $(\mathfrak{h}, \Pi, \Pi^\vee)$ , where  $\mathfrak{h}$  is a  $2n - l$  dimensional complex vector space,

$\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$  and  $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$  are linearly independent subsets in  $\mathfrak{h}^*$  and  $\mathfrak{h}$ , respectively, satisfying

$$\alpha_j(\alpha_i^\vee) = a_{ij} \quad \text{for } i, j = 1, \dots, n.$$

Let  $\mathfrak{g}(A)$  be the Kac-Moody algebra associated to  $A$ . One can see from [5] that  $\mathfrak{g}(A)$  is a complex Lie algebra generated by  $\mathfrak{h}, e_1, \dots, e_n, f_1, \dots, f_n$  with following defining relations:

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} \alpha_i^\vee, & (i, j = 1, \dots, n); \\ [h, h'] &= 0, & (h, h' \in \mathfrak{h}); \\ [h, e_i] &= \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i, & (i = 1, \dots, n; h \in \mathfrak{h}); \\ (\text{ade}_i)^{1-a_{ij}} e_j &= (\text{ad } f_i)^{1-a_{ij}} f_j = 0, & (i \neq j). \end{aligned}$$

Let  $Q = \sum_{i=1}^n \mathbb{Z} \alpha_i$  be the root lattice, and  $Q^+ = \sum_{i=1}^n \mathbb{N} \alpha_i$ . Then

$$\mathfrak{g}(A) = \left( \bigoplus_{\alpha \in Q^+, \alpha \neq 0} \mathfrak{g}_{-\alpha} \right) \oplus \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in Q^+, \alpha \neq 0} \mathfrak{g}_\alpha \right)$$

is a  $Q$ -graded Lie algebra related to Cartan subalgebra  $\mathfrak{h}$ . Here,  $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$  is a root space attached to  $\alpha$ . For  $\alpha > 0$  (resp.  $\alpha < 0$ ),  $\mathfrak{g}_\alpha$  is the linear span of the elements of the form

$$[e_{i_1}, e_{i_2}, \dots, e_{i_s}] \quad (\text{resp. } [f_{i_1}, f_{i_2}, \dots, f_{i_s}])$$

such that  $\alpha_{i_1} + \dots + \alpha_{i_s} = \alpha$  (resp.  $= -\alpha$ ). Moreover, we have

- (i)  $\mathfrak{g}_{\alpha_i} = \mathbb{C} e_i, \quad \mathfrak{g}_{-\alpha_i} = \mathbb{C} f_i,$  for  $i \in \{1, \dots, n\}$ ;
- (ii)  $\mathfrak{g}_{s\alpha_i} = \{0\},$  for  $i \in \{1, \dots, n\}, |s| > 1$ ;
- (iii)  $[e_i, f_i, e_i] = 2e_i \neq 0, \quad [f_i, e_i, f_i] = 2f_i \neq 0,$  for  $i \in \{1, \dots, n\}$ ;
- (iv)  $[e_{i_1}, e_{i_2}, \dots, e_{i_s}] = [e_{i_1}, [e_{i_2}, \dots, e_{i_s}]],$   
 $[f_{i_1}, f_{i_2}, \dots, f_{i_s}] = [f_{i_1}, [f_{i_2}, \dots, f_{i_s}]],$  for  $i_1, \dots, i_s \in \{1, \dots, n\}, s \geq 2$ .

Let  $\mathfrak{b}^\pm = \bigoplus_{\alpha \in Q^+} \mathfrak{g}_{\pm\alpha}$ .  $\mathfrak{b}^+, \mathfrak{b}^-$  are Borel subalgebras of  $\mathfrak{g}(A)$ . One can check that  $\mathfrak{g}(A)$  and  $\mathfrak{b}^\pm$  are finitely generated Lie algebras graded by a nontrivial finite dimensional Cartan subalgebra  $\mathfrak{h}$  and satisfy property (P). Using theorem 3.2, we have corollary 4.3.1.

**Corollary 4.3.1** Let  $A = (a_{ij})_{i,j=1}^n$  be a generalized Cartan matrix,  $\mathfrak{g}(A)$  and  $\mathfrak{b}^\pm$  as above. For  $N \geq 3$ ,

$$\text{Der}^{(N)}(\mathfrak{g}(A)) = \text{Der}(\mathfrak{g}(A)), \quad \text{Der}^{(N)}(\mathfrak{b}^\pm) = \text{Der}(\mathfrak{b}^\pm).$$

**Remark 4.3.2** Let  $\mathfrak{g}$  be a semisimple complex Lie algebra, and  $A$  be the Cartan matrix of  $\mathfrak{g}$ . Then  $\mathfrak{g}(A) = \mathfrak{g}$ . Since any derivation of  $\mathfrak{g}$  is an inner derivation, by corollary 4.3.1, any  $N$ -derivation of semisimple complex Lie algebra is an inner derivation for  $N \geq 3$ .



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